

DIAMETER AND SPECTRAL GAP FOR PLANAR GRAPHS

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ABSTRACT. We prove that the spectral gap of a finite planar graph X is bounded by $\lambda_1(X) \leq C \left(\frac{\log(\text{diam } X)}{\text{diam } X} \right)^2$ where C depends only on the degree of X . We then give a sequence of such graphs showing the the above estimate cannot be improved. This yields a negative answer to a question of Benjamini and Curien on the mixing time of the simple random walk on planar graphs.

1.

In this note we investigate the relationship between the diameters $\text{diam } X$ of finite planar graphs X and their spectral gaps, i.e. the first non-zero eigenvalues $\lambda_1(X)$ of the associated combinatorial Laplacian. To avoid trivial cases, we will only consider connected graphs with $\text{diam } X \geq 2$. Our first result is the following upper bound:

Theorem 1.1. *For every d there is C_d with*

$$\lambda_1(X) \leq C_d \left(\frac{\log(\text{diam } X)}{\text{diam } X} \right)^2$$

for every finite connected planar graph X with degree at most d .

Theorem 1.1 is not so surprising given that Spielman and Teng proved in [8] (see also [4]) that $\lambda_1(X)$ is bounded from above, up to a constant depending on the degree d of X , by the reciprocal of the number of vertices:

$$(1.1) \quad \lambda_1(X) \leq C_d \frac{1}{\text{vol } X}$$

In fact, Theorem 1.1 follows easily from (1.1) and a simple computation. What may be more surprising, and this is the bulk of this paper, is that the bound provided by Theorem 1.1 cannot be improved:

Theorem 1.2. *There are positive numbers d and C and a sequence (X_n) of pairwise distinct triangulations of \mathbb{S}^2 of degree at most d such that*

$$\lambda_1(X_n) \geq C \left(\frac{\log(\text{diam } X_n)}{\text{diam } X_n} \right)^2$$

for all n .

Remark. Notice that (1.1) implies that for the graphs X_n provided by Theorem 1.2 we have $\text{vol } X_n < (\text{diam } X_n)^2$ for all n large enough.

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Recall the following well-known relation between $\lambda_1(X)$ and the *mixing time* $\tau(X)$ of the simple random walk on X :

$$(1.2) \quad \frac{1}{C\lambda_1(X)} \leq \tau(X) \leq C \frac{\log(\text{vol } X)}{\lambda_1(X)}$$

Here C is a constant which depends only on the maximal degree of X . We refer to the reader to [5] for the definition of and facts on mixing times; see Theorem 12.3 and Theorem 12.4 in [5] for a proof of (1.2).

In the light of (1.2) we can translate the upper bound on $\lambda_1(X)$ provided by Theorem 1.1 to a lower bound on the mixing time:

Corollary 1.3. *For every d there is C_d positive with*

$$\tau(X) \geq C_d \left(\frac{\text{diam } X}{\log(\text{diam } X)} \right)^2$$

for every finite connected planar graph X with degree at most d . \square

Similarly we obtain from Theorem 1.2, the remark after the statement of the theorem, and the right inequality in (1.2) that:

Corollary 1.4. *There are d and C positive and a sequence (X_n) of pairwise distinct triangulations of \mathbb{S}^2 with degree at most d and such that*

$$\tau(X_n) \leq C \frac{(\text{diam } X_n)^2}{\log(\text{diam } X_n)}$$

for all n . \square

Corollary 1.4 yields a negative answer to a question asked by Benjamini and Curien [1, Question 8].

We now sketch the idea of the proof of Theorem 1.2. Given an expander (Z_n) we consider, for each n , a graph Y_n obtained by subdividing each edge of Z_n into roughly n^{10} edges. For each n choose a point $p_n \in Y_n$ and let A_n be a rotationally symmetric metric cylinder in which the t -th circle has length roughly equal to the number of points in Y_n which are at distance t from p_n . We prove that, under Neumann boundary conditions, A_n satisfies the desired bound on the spectral gap and we deduce from the work of Mantuano [7] that this implies that any discretization of A_n does too. The desired graphs X_n are constructed by completing triangulations of the annulus A_n to triangulations of the sphere.

Before concluding this introduction we would like to point out that Mantuano's work [7] yields translations of Theorem 1.1 and Theorem 1.2 to the Riemannian setting. In the same spirit we wish to point out that (1.1) follows, again via [7], from the classical Yang-Yau theorem [9].

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Notation: Suppose that (a_n) and (b_n) are sequences of numbers. Throughout this paper we write $a_n \lesssim b_n$ (resp. $a_n \gtrsim b_n$) if there is $C > 0$ with $a_n < Cb_n$ (resp. $a_n > Cb_n$) for all n . If $a_n \lesssim b_n$ and $a_n \gtrsim b_n$, then we write $a_n \approx b_n$.

2.

In this section we recall a few facts about Laplacians on graphs and Riemannian manifolds. We refer the reader [2, 3, 6] for details.

2.1. Graphs will be denoted by capital letters X, Y, Z, \dots , often with super or subscripts. We denote by $V(X)$ the set of vertices and by $E(X)$ the set of edges of a graph X . We indicate by $x \sim y$ that two vertices of X are joined by an edge. The valence, or degree, of a vertex $x \in V(X)$ is the number $\deg(x)$ of edges adjacent to x . The degree of a graph is the maximum of the degrees of its vertices. We assume, often implicitly, that all graphs are connected.

The (combinatorial) Laplacian

$$\Delta_X : \mathbb{R}^{V(X)} \rightarrow \mathbb{R}^{V(X)}$$

of a finite graph X is the linear operator defined by

$$\Delta_X(f)(x) = \sum_{y \sim x} (f(x) - f(y))$$

where we think of elements in $\mathbb{R}^{V(X)}$ as functions on the finite set $V(X)$. With respect to the standard scalar product on $\mathbb{R}^{V(X)}$, Δ_X is symmetric and positive semi definite and hence diagonalizable with spectrum

$$0 = \lambda_0(X) < \lambda_1(X) \leq \lambda_2(X) \leq \dots$$

In this note we are only interested in the smallest positive eigenvalue $\lambda_1(X)$; this quantity is the *spectral gap* of the graph X .

The *Rayleigh quotient* $\mathcal{R}_X(f)$ of $f \in \mathbb{R}^{V(X)}$ is defined to be

$$(2.1) \quad \mathcal{R}_X(f) = \frac{1}{2} \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_{x \in V(X)} f(x)^2}$$

Denoting by \mathcal{W} the space of all 2-dimensional linear subspaces of $\mathbb{R}^{V(X)}$ we obtain the following characterization of $\lambda_1(X)$:

$$\lambda_1(X) = \min_{W \in \mathcal{W}} \max_{f \in W} \mathcal{R}_X(f)$$

This is the simplest incarnation of the so-called *minimax principle*.

Remark. Let $f_1, f_2 \in \mathbb{R}^{V(X)}$ be non-zero elements and let $W \in \mathcal{W}$ be their span. If f_1, f_2 have, when considered as functions, disjoint supports then

$$\max_{f \in W} \mathcal{R}_X(f) = \max\{\mathcal{R}_X(f_1), \mathcal{R}_X(f_2)\}$$

2.2. So far we have considered graphs as purely combinatorial objects, but it will be useful to consider them also as 1-dimensional metric objects. To do so we identify each edge with the unit length interval $[0, 1]$ and consider the induced inner distance on X . Associated to this distance we have the 1-dimensional Hausdorff measure λ_X on X ; this is just an arrogant way of referring to Lebesgue measure on edges. We denote by $\text{Lip}(X)$ the set of Lipschitz functions $X \rightarrow \mathbb{R}$ and identify $\mathbb{R}^{V(X)}$ with the subset of $\text{Lip}(X)$ consisting of functions which are linear on edges.

Given an edge $e \simeq [0, 1]$ and a point x in the interior of e we denote by $\nabla f(x)$ the gradient of $f \in \text{Lip}(X)$; it is defined almost everywhere by Rademacher's theorem. The Rayleigh quotient of $f \in \text{Lip}(X)$ is defined to be:

$$(2.2) \quad \mathcal{R}_X(f) = \frac{\int_X \|\nabla f(x)\|^2 d\lambda_X(x)}{\int_X f(x)^2 d\lambda_X(x)}$$

Letting \mathcal{W} now denote the space of all 2-dimensional linear subspaces of $\text{Lip}(X)$ we have again that

$$\lambda_1(X) = \min_{W \in \mathcal{W}} \max_{f \in W} \mathcal{R}_X(f)$$

Remark. The factor of $\frac{1}{2}$ in (2.1) is due to the fact that every edge is double counted.

We remind the reader that a map

$$f : (X, d_X) \rightarrow (Y, d_Y)$$

between two metric spaces is an L -quasi-isometry if $f(X)$ is L -dense in Y and if for all $x, x' \in X$ we have

$$\frac{1}{L}d_X(x, x') - L \leq d_Y(f(x), f(x')) \leq Ld_X(x, x') + L$$

Two metric spaces are L -quasi-isometric if there is an L -quasi-isometry between them. Quasi-isometric graphs have comparable spectral gaps [3]:

Theorem. *For all d and L there is a constant C such that the following holds: If X and Y are two L -quasi-isometric graphs of degree at most d , then*

$$\frac{1}{C}\lambda_1(X) \leq \lambda_1(Y) \leq C\lambda_1(X)$$

2.3. Suppose that M is a compact Riemannian manifold with possibly non-empty totally geodesic boundary ∂M , and recall that in the presence of boundary we say that a function on f is smooth if it admits a smooth extension to some manifold with empty boundary containing M . The Laplacian of a function $f \in C^\infty(M)$ is defined to be

$$\Delta(f) = -\text{div} \nabla(f)$$

where div stands for the divergence and ∇ for the gradient. The Rayleigh quotient of $f \in C^\infty(M)$ is again defined by

$$\mathcal{R}_M(f) = \frac{\int_M \|\nabla f(x)\|^2 d\text{vol}_M(x)}{\int_M f(x)^2 d\text{vol}_M(x)}$$

where $d\text{vol}_M$ is the volume form of M .

Denoting by ν the inward normal vector field along ∂M we have by Green's theorem that

$$(2.3) \quad \int_M f \Delta g d\text{vol} - \int_M \langle \nabla f, \nabla g \rangle d\text{vol} = \int_{\partial M} f \langle \nabla g, \nu \rangle d\text{vol}$$

for all $f, g \in C^\infty(M)$; here $\langle \cdot, \cdot \rangle$ stands for the Riemannian metric.

Denote by $C_N^\infty(M)$ the space of smooth functions $g \in C^\infty(M)$ satisfying Neumann's boundary condition $\langle \nabla g, \nu \rangle = 0$. It follows from (2.3) that the

restriction of the Laplacian to $C_N^\infty(M)$ is a self-adjoint operator. Moreover, its spectrum is discrete and non-negative

$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \dots$$

In particular, $L^2(M)$ admits a Hilbert basis $(\phi_i)_i \subset C_N^\infty(M)$ consisting of eigenfunctions of Δ :

$$\Delta(\phi_i) = \lambda_i(M)\phi_i$$

Green's theorem (2.3) implies that if $f \in C_N^\infty(M)$ is a $\lambda_i(M)$ eigenfunction, then $\lambda_1(M) = \mathcal{R}_M(f)$.

As it is the case for graphs, the smallest positive eigenvalue $\lambda_1(M)$ can be again computed via the minimax principle

$$(2.4) \quad \lambda_1(M) = \min_{W \in \mathcal{W}_N} \max_{f \in W} \mathcal{R}_M(f)$$

where this time \mathcal{W}_N is the set of all 2-dimensional linear subspaces of $C_N^\infty(M)$. We derive a second version of the minimax principle:

Lemma 2.1. *Suppose that M is a compact Riemannian manifold with totally geodesic boundary and let $\lambda_1(M)$ be the first non-zero eigenvalue of the Laplacian on M with Neumann boundary conditions. Then*

$$\lambda_1(M) = \min_{W \in \mathcal{W}} \max_{f \in W} \mathcal{R}_M(f)$$

where \mathcal{W} is the set of all 2-dimensional linear subspaces of $C^\infty(M)$.

Proof. Let M' be the double of M , $\sigma : M' \rightarrow M'$ the isometric involution with $M'/\sigma = M$, and $\pi : M' \rightarrow M$ the associated orbit map. For $f \in C^\infty(M)$, the function $f' = f \circ \pi$ is Lipschitz and hence is the uniform limit of a sequence of smooth functions $f'_n \in C^\infty(M')$ such that

$$(2.5) \quad \lim_{n \rightarrow \infty} \|\nabla f' - \nabla f'_n\| = 0 \text{ in } L^2(M')$$

Replacing f'_n by $\frac{1}{2}(f'_n + f'_n \circ \sigma)$ we may assume that the functions f'_n are σ invariant and hence descend under π to smooth functions $f_n \in C_N^\infty(M)$. From (2.5) we obtain that

$$\lim_{n \rightarrow \infty} \mathcal{R}_M(f_n) = \mathcal{R}_M(f)$$

The claim follows now from (2.4). \square

2.4. We say that a closed Riemannian manifold M has d -bounded geometry if

- M has at most dimension d ,
- the sectional curvature is pinched by $|\kappa_M| \leq d$, and
- the length of the shortest closed geodesic is bounded from below by $\frac{1}{d}$.

If M is a compact manifold with totally geodesic boundary then we say that M has d -bounded geometry if its double does.

Remark. Recall that if M is closed then an upper bound on the sectional curvature and a lower bound on the length of the shortest closed geodesic yield a lower bound on the injectivity radius.

Discretizations of Riemannian manifolds have often been used in the literature (see e.g. [3]). The following is a well-known fact:

Lemma 2.2. *For every d there are L and d' such that every compact Riemannian manifold M with d -bounded geometry admits a triangulation T whose 1-skeleton $T^{(1)}$ is a graph of valence at most d' which is L -quasi-isometric to M . \square*

Abusing notation we will from now on make no distinction between triangulations and their 1-skeleta.

To prove Theorem 1.2 we will use the fact that quasi-isometric graphs and manifolds with bounded geometry have comparable spectral gaps. In [7], Mantuano proved such a result for closed manifolds via the minimax principle and a comparison of Rayleigh quotients of functions on the manifold and on the graph. The arguments used to compare Rayleigh quotients go through without any additional work in the presence of boundary. Moreover, since by Lemma 2.1 the first eigenvalue of the Laplacian with Neumann boundary conditions can be computed applying the minimax principle to *all* smooth functions on the manifold, we see that Mantuano's theorem applies also to manifolds with boundary:

Theorem (Mantuano [7]). *For all d and L there is a constant C such that the following holds: Suppose that X is a graph with valence at most d , M is a compact Riemannian manifold with d -bounded geometry and possibly non-empty boundary, and that X and M are L -quasi-isometric to each other, then*

$$\frac{1}{C}\lambda_1(X) \leq \lambda_1(M) \leq C\lambda_1(X)$$

where $\lambda_1(X)$ and $\lambda_1(M)$ are the first non-zero eigenvalues of the combinatorial Laplacian on X and respectively the Laplacian with Neumann boundary conditions on M .

3.

In this section we prove Theorem 1.1; we also introduce some notation used later on.

3.1. Suppose that X is a finite graph considered as a 1-dimensional metric space. For $p \in X$ we have the distance function

$$(3.1) \quad \delta_p : X \rightarrow \mathbb{R}, \quad x \mapsto d(p, x)$$

with range $[0, \text{diam}_p(X)]$ where

$$(3.2) \quad \text{diam}_p(X) = \max\{d(p, x) \mid x \in X\}$$

Notice that $\frac{1}{2} \text{diam}(X) \leq \text{diam}_p(X) \leq \text{diam}(X)$.

The measure $(\delta_p)_*(\lambda_X)$ obtained by pushing the Lebesgue measure λ_X of X forward via δ_p is absolutely continuous with respect to the Lebesgue measure. We denote by

$$\rho_{X,p} : \mathbb{R} \rightarrow \mathbb{R}^+$$

the Radon-Nicodym derivative of $(\delta_p)_*(\lambda_X)$ with respect to dt , meaning that for all continuous functions f we have

$$(3.3) \quad \int_{\mathbb{R}} f(t) \rho_{X,p}(t) dt = \int_X (f \circ \delta_p)(x) d\lambda_X(x)$$

Notice that $\rho_{X,p}$ takes integer values and that $\rho_{X,p}(t) \geq 1$ for all $t \in [0, \text{diam}_p(X)]$ and $\rho_{X,p}(t) = 0$ otherwise. We state a useful observation:

Lemma 3.1. *Suppose that X is a finite graph, $p \in X$ is a base point and*

$$F : [0, \text{diam}_p(X)] \rightarrow \mathbb{R}$$

is a Lipschitz function, then

$$\mathcal{R}_X(F \circ \delta_p) = \frac{\int_0^{\text{diam}_p(X)} F'(t)^2 \rho_{X,p}(t) dt}{\int_0^{\text{diam}_p(X)} F(t)^2 \rho_{X,p}(t) dt}$$

The proof of Lemma 3.1 is elementary and we leave it to the reader. \square

3.2. Armed with the notation we just introduced, we prove the following general fact about graphs:

Proposition 3.2. *For all V and r there is C such that if X is a finite graph satisfying $\text{vol } X \leq V(\text{diam } X)^r$, then*

$$\lambda_1(X) \leq C \left(\frac{\log(\text{diam } X)}{\text{diam } X} \right)^2$$

Proof. Let $k \in \mathbb{Z}$ be the integer part of $\log \frac{\text{diam}(X)}{2}$ and notice that this implies that if $d(p_1, p_2) = \text{diam}(X)$ then the balls $B(p_i, e^k)$ in X of radius e^k centered at p_1 and p_2 are disjoint. For $i = 1, 2$ we are going to construct, as long as the diameter of X is over some threshold depending on V and r , a function $f_i \in C^\infty(X)$ supported by $B(p_i, e^k)$ and with Rayleigh quotient

$$\mathcal{R}_X(f_i) \leq (1 + \log(r + 2)) \frac{k}{e^k}$$

Once this is done, the desired claim follows from the choice of k because as we noticed earlier

$$\lambda_1(X) \leq \max\{\mathcal{R}_X(f_1), \mathcal{R}_X(f_2)\}$$

For the sake of concreteness assume that $i = 1$ and set $p = p_1$. Consider the distance function

$$\delta = \delta_p : X \rightarrow [0, \text{diam}(X)]$$

and let $\rho = \rho_{X,p}$ be the function satisfying (3.3). Notice that

$$\int_0^{e^k} \rho(t) dt = \text{vol}(B(p, e^k)) \leq eV e^{rk}$$

by the choice of k and the assumption on the volume of X . Divide the interval $[0, e^k]$ into k consecutive intervals E_1, \dots, E_k of equal length $\frac{e^k}{k}$; set also $E_0 = \emptyset$. Notice that

$$\int_{E_1} \rho(t) dt = \text{vol}(\delta^{-1}(E_1)) \geq \frac{e^k}{k}$$

and suppose that for some L we have

$$\int_{E_{j+1}} \rho(t) dt \geq L \int_{E_j} \rho(t) dt$$

for all $j = 1, \dots, k-1$. Then we get from the bound on $\text{vol } X$ that

$$eVe^{rk} \geq \text{vol}(\delta^{-1}(E_k)) = \int_{E_k} \rho(t)dt \geq L^{k-1} \int_{E_1} \rho(t)dt \geq L^{k-1} \frac{e^k}{k}$$

It follows that (with finitely many graphs X as possible exceptions) we have $L < \log(r+2)$. In particular, as long as $\text{diam}(X)$ is over some threshold, there is $j \in \{1, \dots, k-1\}$ with

$$(3.4) \quad \int_{E_{j-1}} \rho(t)dt + \int_{E_{j+1}} \rho(t)dt \leq (1 + \log(r+2)) \int_{E_j} \rho(t)dt$$

Consider now the Lipschitz function $F : [0, \text{diam}(X)] \rightarrow \mathbb{R}$ given by

$$\begin{aligned} F(t) &= 0 && \text{if } t \notin E_{j-1} \cup E_j \cup E_{j+1} \\ F(t) &= t - \frac{(j-1)e^k}{k} && \text{if } t \in E_{j-1} \\ F(t) &= \frac{e^k}{k} && \text{if } t \in E_j \\ F(t) &= \frac{(j+2)e^k}{k} - t && \text{if } t \in E_{j+1} \end{aligned}$$

and notice that

$$\begin{aligned} F'(t) &= 0 && \text{if } t \notin E_{j-1} \cup E_j \cup E_{j+1} \\ |F'(t)| &= 1 && \text{if } t \in E_{j-1} \cup E_j \cup E_{j+1} \end{aligned}$$

The function $F \circ \delta$ is Lipschitz, supported by $B(p, e^k)$ and satisfies

$$(3.5) \quad \int_0^{\text{diam}(X)} F'(t)^2 \rho(t)dt = \int_{E_{j-1}} \rho(t)dt + \int_{E_{j+1}} \rho(t)dt$$

On the other hand, we have

$$(3.6) \quad \int_0^{\text{diam}(X)} F(t)^2 \rho(t)dt \geq \int_{E_j} f(t)^2 \rho(t)dt = \frac{e^k}{k} \int_{E_j} \rho(t)dt$$

Combining Lemma 3.1 with equaltions (3.4), (3.5) and (3.6) we obtain for $f = F \circ \delta$ that

$$\mathcal{R}_X(f) = \mathcal{R}_X(F \circ \delta) \leq (1 + \log(r+2)) \frac{k}{e^k}$$

Which is what we needed to prove. □

3.3. Spielman and Teng [8] (see also [4]) proved:

Theorem (Spielman-Teng). *For every d there is C_d with*

$$\lambda_1(X) \leq C_d \frac{1}{\text{vol } X}$$

for every finite planar graph X of degree at most d .

Theorem 1.1 follows immediatelly from the Spielman-Teng theorem and Proposition 3.2.:

Theorem 1.1. *For every d there is C_d with*

$$\lambda_1(X) \leq C_d \left(\frac{\log(\text{diam } X)}{\text{diam } X} \right)^2$$

for every finite connected planar graph X with degree at most d . \square

4.

In section 6 we will prove:

Proposition 4.1. *There is d and a sequence A_n of Riemannian surfaces homeomorphic to $\mathbb{S}^1 \times [0, 1]$, with totally geodesic boundary and d -bounded geometry, and such that:*

$$\text{vol}(A_n) \approx n^{11}, \quad \text{diam}(A_n) \approx n^{10} \log(n), \quad \lambda_1(A_n) \approx n^{-20}$$

Moreover, each component of ∂A_n has unit length.

Recall that $\lambda_1(A_n)$ is the first positive eigenvalue of the Laplacian on A_n with Neumann boundary conditions.

Assuming Proposition 4.1, we prove now Theorem 1.2:

Theorem 1.2. *There are d and C positive and a sequence (X_n) of pairwise distinct triangulations of \mathbb{S}^2 with degree at most d and such that*

$$\lambda_1(X_n) \geq C \left(\frac{\log(\text{diam } X_n)}{\text{diam } X_n} \right)^2$$

for all n .

Proof. Let A_n be Riemannian surfaces provided by Proposition 4.1, and for each n let T_n be a triangulation of A_n as provided by Lemma 2.2. Denoting by D_1, D_2 the cycles in T_n corresponding to the boundary components of A_n , let X_n be obtained from T_n by coning off D_1 and D_2 , each one of them to a different point. By construction X_n is a triangulation of \mathbb{S}^2 . To see that X_n has uniformly bounded degree observe that T_n does by Lemma 2.2 and that D_1 and D_2 have uniformly bounded many edges because both boundary components of A_n have unit length. This also shows that T_n and X_n are uniformly quasi-isometric to each other. Since T_n and Σ_n are uniformly quasi-isometric to each other, we obtain

$$(4.1) \quad \text{diam } X_n \approx \text{diam } T_n \approx \text{diam } A_n \approx n^{10} \log(n)$$

$$(4.2) \quad \lambda_1(X_n) \approx \lambda_1(T_n) \approx \lambda_1(\Sigma_n) \approx n^{-20}$$

It follows from (4.1) and (4.2) that there is some C with

$$\lambda_1(X_n) \geq C \left(\frac{\log(\text{diam } X_n)}{\text{diam } X_n} \right)^2$$

for all n , as we wanted to prove. \square

5.

In this section we construct a family of graphs Y_n needed in the proof of Proposition 4.1. We stress that the graphs Y_n are not planar.

Lemma 5.1. *There is a sequence (Y_n, p_n) of rooted graphs with vertices of degree at most 3, with*

$$\text{vol } Y_n \approx n^{11}, \quad \text{diam } Y_n \approx \log(n)n^{10}, \quad \lambda_1(Y_n) \approx n^{-20}$$

and such that the function $\rho_{Y_n, p_n} : \mathbb{R} \rightarrow \mathbb{R}_+$ satisfying (3.3) has the following properties:

- (1) *For all $t \in \mathbb{R}$ we have $\rho_{Y_n, p_n}(t) \lesssim n$.*
- (2) *For any two $t, s \in \mathbb{R}$ with $|t - s| < \frac{1}{2}$ we have $|\rho_{Y_n, p_n}(t) - \rho_{Y_n, p_n}(s)| \leq 3$.*

We explain briefly the meanings of the numbered statements in Lemma 5.1: (1) asserts that the cardinality of the distance sphere $\delta_{Y_n, p_n} = t$ in Y_n centered at p_n is bounded from above by some multiple of n , and (2) asserts that the number of points in the distance spheres $\delta_{Y_n, p_n} = t$ and $\delta_{Y_n, p_n} = t + h$ jumps up or down by at most 3 if $h < \frac{1}{2}$.

We define some terms used in the proof of Lemma 5.1. Given a rooted graph (Y, p) we say that $t \in [0, \text{diam}_p Y]$ is a *critical value* of the distance function $\delta_p : Y \rightarrow [0, \text{diam}_p Y]$ if the density $\rho_{Y, p} : \mathbb{R} \rightarrow \mathbb{R}_+$ is not continuous at t . Notice that 0 and $\text{diam}_p Y$ are critical values and that all critical values belong to $\frac{1}{2}\mathbb{Z}$, and recall that $\rho_{Y, p}$ takes values in \mathbb{N} . A critical value t is *good* if the step of discontinuity of $\rho_{Y, p}$ at t is at most 3. The critical value 0 is good if and only if $\deg(p) \leq 3$. Claim (2) of Lemma 5.1 is that all critical values of the distance function associated to (Y_n, p_n) are good.

Obviously the choice of the number 3 in the definition of *good critical point* is somewhat arbitrary. It is tailored to the fact that we will work with graphs of valence at most 3. Basically, one should think of good critical values in terms of general position; we hope that figure 1 makes this remark clear.

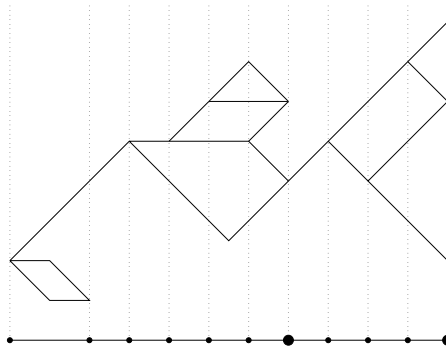


FIGURE 1. The distance function is the projection to the x -axis. The dots are the critical values; the good ones small and the bad ones large.

Remark. Suppose that Y is a finite graph of degree at most 3, with no vertices of valence 1, and that $p \in V(p)$ is a vertex. Both the number of critical points of $\delta_p : Y \rightarrow [0, \text{diam}_p Y]$ and $\max_{t \in [0, \text{diam}_p Y]} \rho_{Y,p}(t)$ are bounded by four times the number of trivalent vertices plus 2.

Proof of Lemma 5.1. Let (Z_n) be an expander family consisting of trivalent graphs without vertices of valence 1 and with $\text{vol } Z_n \approx n$. That the sequence (Z_n) is an expander means that there is $\epsilon > 0$ with $\lambda_1(Z_n) \geq \epsilon$ for all n . It is known that this implies that $\text{diam}(Z_n) \approx \log(n)$. In this section, every graph with an n for subscript is going to be homeomorphic to Z_n . In particular, (1) in Lemma 5.1 will be automatically satisfied by the remark above.

For each n let \hat{Y}_n be the graph obtained by subdividing each edge in n^{10} edges. Metrically, this amounts to scaling the graph Z_n by a factor n^{10} . It follows that

$$\text{vol } \hat{Y}_n \approx n^{11}, \quad \lambda_1(\hat{Y}_n) \geq \frac{\epsilon}{n^{-20}}, \quad \text{diam } \hat{Y}_n \approx \log(n)n^{10}$$

Choose now points $\hat{p}_n \in \hat{Y}_n$. It is easy to see that not all the critical values of the distance function $\delta_{\hat{p}_n} : \hat{Y}_n \rightarrow [0, \text{diam}_{\hat{p}_n} \hat{Y}_n]$ are good, meaning that (\hat{Y}_n, \hat{p}_n) does not satisfy (2) in the statement of Lemma 5.1. We are going to inductively perturb \hat{Y}_n and construct

$$(\hat{Y}_n^0, \hat{p}_n^0) = (\hat{Y}_n, \hat{p}_n), (\hat{Y}_n^1, \hat{p}_n^1), (\hat{Y}_n^2, \hat{p}_n^2), \dots$$

so that in every step more and more critical values are good.

Suppose that we have constructed $(\hat{Y}_n^i, \hat{p}_n^i)$ with \hat{Y}_n^i homeomorphic to Z_n , suppose that t is the smallest bad critical value of $\delta_{\hat{p}_n^i} : \hat{Y}_n^i \rightarrow \mathbb{R}$ and notice that $t > 0$. Choose a point $x \in \hat{Y}_n^i$ at distance $\delta_{\hat{p}_n^i}(x) = t$ of the base point so that either x is a trivalent vertex or a local maximum of $\delta_{\hat{p}_n^i}$. For each other point $y \in \hat{Y}_n^i$ with $\delta_{\hat{p}_n^i}(y) = t$ subdivide each edge adjacent to y into two edges and let \hat{Y}_n^{i+1} be the so obtained graph rooted at the vertex \hat{p}_n^{i+1} corresponding to \hat{p}_n^i . Notice that $\rho_{\hat{p}_n^{i+1}}(s) = \rho_{\hat{p}_n^i}(s)$ for all $s < t$, that t is still a critical value of $\rho_{\hat{p}_n^{i+1}}$, but that this time it is a good one. It follows that $\rho_{\hat{p}_n^{i+1}}$ has at least i good critical values. Since \hat{Y}_n^{i+1} is homeomorphic to Z_n it follows that the total number of critical points is bounded by $4 \text{vol } Z_n + 2$ and hence we have to repeat the process $N \leq 4 \text{vol } Z_n + 2$ times to end up with a graph

$$(Y_n, p_n) = (\hat{Y}_n^N, \hat{p}_n^N)$$

for which the function $\delta_{p_n} : Y_n \rightarrow [0, \text{diam}_{p_n} Y_n]$ has only good critical points; in particular (2) in Lemma 5.1 holds. Moreover, since the graph Y_n has been obtained from Z_n by subdividing each edge somewhere between n^{10} times and $n^{10} + 4 \text{vol } Z_n + 2$ times, we still have

$$\text{vol } Y_n \approx n^{11}, \quad \lambda_1(Y_n) \approx n^{-20}, \quad \text{diam } Y_n \approx \log(n)n^{10}$$

The validity of (1) follows from the fact that Y_n is homeomorphic to Z_n for all n . \square

6.

We can now prove Proposition 4.1:

Proposition 4.1. *There is d and a sequence A_n of Riemannian surfaces homeomorphic to $\mathbb{S}^1 \times [0, 1]$, with totally geodesic boundary and d -bounded geometry, and such that:*

$$\text{vol}(A_n) \approx n^{11}, \quad \text{diam}(A_n) \approx n^{10} \log(n), \quad \lambda_1(A_n) \approx n^{-20}$$

Moreover, each component of ∂A_n has unit length.

Let (Y_n, p_n) be the sequence of rooted graphs provided by Lemma 5.1, set

$$R(n) = \text{diam}_{p_n}(Y_n)$$

and consider the functions

$$\rho_{Y_n, p_n} : \mathbb{R} \rightarrow \mathbb{R}_+$$

satisfying (3.3). The support of ρ_{Y_n, p_n} is $[0, R(n)]$ and recall that for every t therein we have $\rho_{Y_n, p_n}(t) \geq 1$. On the other hand, we have $\rho_{Y_n, p_n}(t) \lesssim n$ by Lemma 5.1 (1).

Lemma 5.1 (2) implies that there is some positive constant C such that for each n there is a smooth function

$$\sigma_n : \mathbb{R} \rightarrow \mathbb{R}_+$$

satisfying:

- (1) $C^{-1} \rho_{Y_n, p_n}(t) \leq \sigma_n(t) \leq C \rho_{Y_n, p_n}(t)$ for all $t \in [0, R(n)]$,
- (2) $|\sigma_n''(t)| \leq C$ for all t , and
- (3) $\sigma(t) = 1$ for all $t \in [0, 1] \cup [R(n) - 1, R(n)]$.

Denote by \mathbb{S}^1 the circle of length 1 and consider the cylinder

$$A_n = ([0, R(n)] \times \mathbb{S}^1, dt^2 + \sigma_n^2(t) d\theta^2)$$

. Each of the two boundary components of ∂A_n is totally geodesic of length 1 and has a collar neighborhood isometric to a product. Observe that there is an isometric action $\mathbb{S}^1 \curvearrowright A_n$ whose orbits agree with the fibers of the projection

$$\pi : A_n \rightarrow [0, R(n)]$$

We claim that the surfaces A_n have uniformly bounded geometry.

Lemma 6.1. *There is d such that A_n has d -bounded geometry for all d .*

Proof. We need to prove that the double Σ_n of A_n has d -bounded geometry for some d and all n . To do this we need to estimate the sectional curvature of Σ_n and the length of the shortest closed geodesic there in. Notice that by symmetry it suffices to bound the sectional curvature of A_n at $(t, 0)$. Since A_n admits an isometric \mathbb{S}^1 -action whose orbit over $t \in [0, R(n)]$ has length $\sigma(t)$, it is classical that the sectional curvature at $(t, 0)$ is given by

$$\kappa_{A_n}(t, \theta) = \frac{\sigma_n''(t)}{\sigma_n(t)}$$

Since $\sigma_n(t) \geq C^{-1}$ and $|\sigma_n''(t)| \leq C$ for all $t \in [0, R(n)]$ we deduce that A_n , and hence Σ_n , has sectional curvature bounded by $|\kappa_{\Sigma_n}| \leq C^2$.

The action $\mathbb{S}^1 \curvearrowright A_n$ extends to an isometric action $\mathbb{S}^1 \curvearrowright \Sigma_n$ with associated Killing vector field $\frac{\partial}{\partial \theta}$. If $\gamma(t) = (T(t), \theta(t))$ is a geodesic in Σ_n then $\langle \gamma'(t), \frac{\partial}{\partial \theta} \rangle$ is constant by Clairaut's theorem. In particular, it follows $t \mapsto \theta(t)$ is monotone and hence that either γ is orthogonal to the orbits of \mathbb{S}^1 or has at least length $\min_t \sigma_n(t) \geq C^{-1}$. Since geodesics orthogonal to \mathbb{S}^1 -orbits have length at least $2R(n)$ we have proved that the surfaces Σ_n have uniformly bounded geometry. \square

We estimate the diameter and volume of A_n .

Lemma 6.2. $\text{vol } A_n \approx n^{11}$ and $\text{diam } A_n \approx \log(n)n^{10}$.

Proof. The Riemannian volume form of A_n is given by $d \text{vol}_{A_n} = \sigma_n(t) dt d\theta$ in the coordinates (t, θ) . In particular we have

$$\text{vol } A_n = \int_0^{R(n)} \sigma_n(t) dt \approx \int_0^{R(n)} \rho_{Y_n, p_n}(t) dt = \text{vol } Y_n \approx n^{11}$$

Here the second to last equality follows from the definition of ρ_{Y_n, p_n} (3.3). The last statement is true by Lemma 5.1.

To estimate the diameter notice that the distance between points in $\pi^{-1}(t)$ is bounded by $\sigma_n(t) \leq C \rho_{Y_n, p_n}(t) \lesssim n$ by assertion (1) in Lemma 5.1. On the other hand A_n contains a geodesic arc of length exactly $R(n) = \text{diam}_{p_n}(Y_n) \simeq \log(n)n^{10}$ intersecting every fiber of π we deduce that

$$\text{diam } A_n \lesssim \log(n)n^{10}$$

Finally, since the projection $\pi : A_n \rightarrow [0, R(n)]$ is 1-Lipschitz it follows that $\text{diam}(A_n) \geq R(n) \gtrsim \log(n)n^{10}$. \square

Recall that $\lambda_1(A_n)$ is the first positive eigenvalue of the Laplacian on A_n with Neumann boundary conditions. In order to prove Proposition 4.1 it remains to bound $\lambda_1(A_n)$; we start with the upper bound.

Lemma 6.3. $\lambda_1(A_n) \lesssim n^{-20}$.

Proof. By Lemma 6.1, the surfaces A_n have d -bounded geometry for some uniform d . In particular, there are by Lemma 2.2 constants d' and L such that for all n the surface A_n admits a triangulation T_n of valence at most d' and which is L -quasi-isometric to A_n . From Lemma 6.2 we obtain that $\text{diam } T_n \approx \log(n)n^{10}$. Since T_n is planar it follows from Theorem 1.1 that

$$\lambda_1(T_n) \lesssim n^{-20}$$

Finally, using again that A_n and T_n are uniformly quasi-isometric to each other, it follows that $\lambda_1(A_n) \approx \lambda_1(T_n)$. \square

It remains to bound $\lambda_1(A_n)$ from below. When doing so we will use the following fact:

Lemma 6.4. *Let $\phi : [0, R] \rightarrow (0, \infty)$ be a smooth function which is constant near the boundary and consider the surface*

$$A = ([0, R] \times \mathbb{S}^1, dt^2 + \phi^2 d\theta^2)$$

If $\lambda_1(A) < \frac{4\pi^2}{\max\{\phi(t)^2 | t \in [0, R]\}}$ then every λ_1 -eigenfunction $f \in C_N^\infty(S)$ is constant on the orbits of the isometric action $\mathbb{S}^1 \curvearrowright A$, $(\eta, (t, \theta)) \mapsto (t, \eta + \theta)$.

Proof. If f is a λ_1 -eigenfunction, then for all $\eta \in \mathbb{S}^1$ the function $f(t, \theta) = f(t, \theta + \eta)$ is also a λ_1 -eigenfunction. In particular, the average

$$\hat{f}(t, \theta) = \int_{\mathbb{S}^1} f(t, \theta + \eta) d\eta$$

along the orbits $S(t)$ of the \mathbb{S}^1 -action is also a λ_1 -eigenfunction. We need to prove that $f = \hat{f}$. Supposing that this is not the case, then $F = f - \hat{f}$ is non-zero, a λ_1 -eigenfunction, and satisfies

$$0 = \int_{S(t)} F(t, \theta) d\theta$$

for all t . Notice that this last equality implies that

$$\frac{4\pi^2}{\phi(t)^2} = \lambda_1(S(t)) \leq \frac{\int_{S(t)} \left\| \frac{\partial F}{\partial \theta}(t, \theta) \right\|^2 d\text{vol}_{S(t)}(\theta)}{\int_{S(t)} F(t, \theta)^2 d\text{vol}_{S(t)}(\theta)}$$

We estimate the Rayleigh quotient of F :

$$\begin{aligned} \int_A \|\nabla F(t, \theta)\|^2 d\text{vol}_A(t, \theta) &\geq \int_A \left\| \frac{\partial F}{\partial \theta}(t, \theta) \right\|^2 d\text{vol}_A(t, \theta) \\ &= \int_0^R \int_{S(t)} \left\| \frac{\partial F}{\partial \theta}(t, \theta) \right\|^2 d\text{vol}_{S(t)}(\theta) dt \\ &\geq \int_0^R \lambda_1(S(t)) \int_{\mathbb{S}^1_t} F(t, \theta)^2 d\text{vol}_{S(t)}(\theta) dt \\ &= \int_0^R \frac{4\pi^2}{\phi(t)^2} \int_{S(t)} F(t, \theta)^2 d\text{vol}_{S(t)}(\theta) dt \\ &\geq \frac{4\pi^2}{\max\{\phi(t)^2 | t \in [0, R]\}} \int_0^R \int_{S(t)} F(t, \theta)^2 d\text{vol}_{S(t)}(\theta) dt \\ &\geq \frac{4\pi^2}{\max\{\phi(t)^2 | t \in [0, R]\}} \int_A F(t, \theta)^2 d\text{vol}_A(t, \theta) \end{aligned}$$

Since F is a λ_1 -eigenfunction, we obtain from the computation above that

$$\lambda_1(A) = \mathcal{R}_A(F) = \frac{\int_A \|\nabla F(t, \theta)\|^2 d\text{vol}_A(t, \theta)}{\int_A F(t, \theta)^2 d\text{vol}_A(t, \theta)} \geq \frac{4\pi^2}{\max\{\phi(t)^2 | t \in [0, R]\}}$$

contradicting our assumption. \square

We are now ready to prove Proposition 4.1:

Proof of Proposition 4.1. Having proved Lemma 6.1, Lemma 6.2 and Lemma 6.3, it remains to bound $\lambda_1(A_n)$ from below.

By construction, the fibers of the projection $\pi : A_n \rightarrow [0, R(n)]$ are the orbits of the isometric circle action $\mathbb{S}^1 \curvearrowright A_n$. Moreover, as we already observed when proving Lemma 6.2, each such fiber has length $\lesssim n$. In particular, we obtain

from Lemma 6.4 that every $\lambda_1(A_n)$ -eigenfunction $f \in C_N^\infty(A_n)$ is constant along the fibers of π . This means that there is a function

$$F : [0, R(n)] \rightarrow \mathbb{R}$$

with $f = F \circ \pi$. Consider the 2-dimensional space $W \subset C^\infty([0, R(n)])$ spanned by the restriction of F to $[0, R(n)]$ and the constant function $\mathbb{1}(x) = 1$. We can summarize much of what we have said so far in the following equation:

$$(6.1) \quad \lambda_1(A_n) = \max\{\mathcal{R}_{A_n}(\phi \circ \pi) \mid \phi \in W\}$$

Recall that on the other hand we have the distance function

$$\delta_{p_n} : Y_n \rightarrow [0, \text{diam}_{p_n}(Y_n)]$$

and by the minimax principle we have

$$(6.2) \quad \lambda_1(Y_n) \leq \max\{\mathcal{R}_{Y_n}(\phi \circ \delta_p) \mid \phi \in W\}$$

We estimate $\mathcal{R}_{A_n}(\phi \circ \pi)$ for $\phi \in W$:

$$\begin{aligned} \int_{A_n} \|\nabla(\phi \circ \pi)\|^2 d\text{vol}_S(t, \theta) &= \int_{A_n} \phi'(t)^2 d\text{vol}_S(t, \theta) \\ &= \int_0^{R(n)} \phi'(t)^2 \sigma_n(t) dt \\ &\geq \frac{1}{C} \int_0^{R(n)} \phi'(t)^2 \rho_{Y_n, p_n}(t) dt \\ \int_{A_n} (\phi \circ \pi)^2 d\text{vol}_S(t, \theta) &= \int_{A_n} \phi(t)^2 d\text{vol}_S(t, \theta) \\ &= \int_0^{R(n)} \phi(t)^2 \sigma_n(t) dt \\ &\leq C \int_0^{R(n)} \phi(t)^2 \rho_{Y_n, p_n}(t) dt \end{aligned}$$

Combining these two inequalities with Lemma 3.1 we obtain

$$\mathcal{R}_{A_n}(\phi \circ \pi) \geq \frac{1}{C^2} \mathcal{R}_{Y_n}(\phi \circ \delta_{p_n})$$

for all $\phi \in W$. In particular (6.1), (6.2) and Lemma 5.1 imply that

$$\lambda_1(A_n) \geq \frac{1}{C^2} \lambda_1(Y_n) \gtrsim n^{-20}$$

Which concludes the proof of Proposition 4.1. \square

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